Asymptotic preserving schemes (AP) for magnetically confined plasmas

Highly anisotropic Vlasov equation

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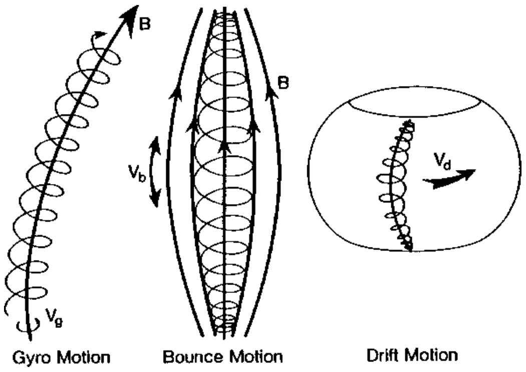
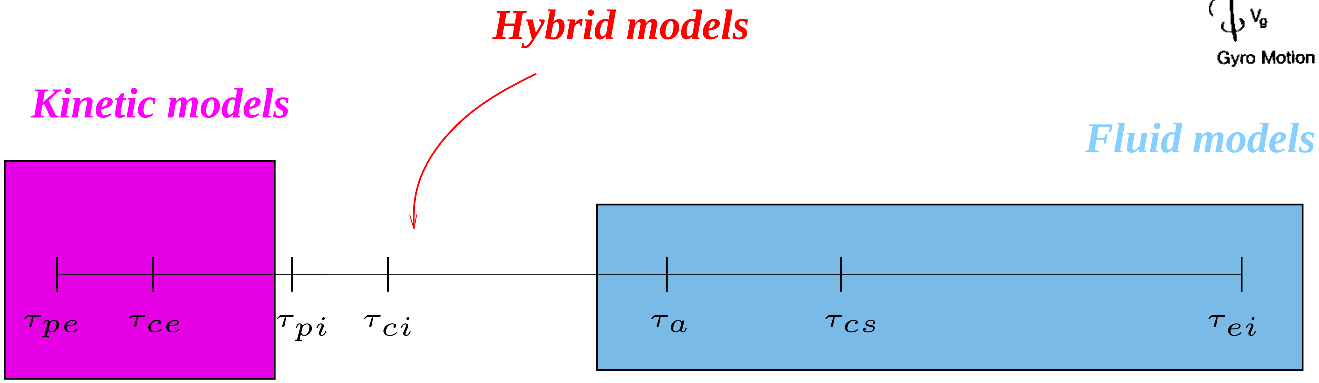
Multi-scale plasma dynamics

Plasma dynamics is characterized by multi-scale phenomena

$\longrightarrow$ Strong magn. fields create anisotropies

$\longrightarrow$ Particles gyrate around the field lines





$\tau_{pe,pi}$: Inv. electr./ion plasma freq.

$\tau_{ce,ci}$: Electr./ion cyclotron period

τ_a: Alfen wave period

τ_{cs}: Ion sound period

τ_{ei}: Electr-ion collision time

λ_D: Debye length

$\rho_{e,i}$: Electr./ion Larmor radius

$\delta_{e,i} = c/\omega_{pe,pi}$: Electr./ion skin depth

$\omega_{pe,pi}$: Electr./ion plasma frequency

c: sound speed

Electron kinetic equation in the adiabatic regime

Work based on:

[1] C. Negulescu, S. Possanner "Closure of the strongly-magnetized electron fluid equations in the adiabatic regime", SIAM Multiscale Model. Simul. 14 (2016), no. 2, 839–873.

Electron-ion kinetic system

Physical context: study of tokamak fusion plasmas (ITER reactor)

Starting model: Boltzmann equations for the two species (ions, electrons)

$$\begin{cases} \partial_t f_i + \mathbf{v} \cdot \nabla_x f_i + \dfrac{q}{m_i} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_i = Q_{ii}(f_i) + Q_{ie}(f_i, f_e) \\ \partial_t f_e + \mathbf{v} \cdot \nabla_x f_e - \dfrac{q}{m_e} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_e = Q_{ee}(f_e) + Q_{ei}(f_e, f_i) \,, \end{cases}$$

coupled to the Poisson equation for the electrostatic potentiel

$$-\Delta \phi = \frac{q}{\varepsilon_0} (n_i - n_e) \,, \quad \mathbf{E} = -\nabla \phi \,.$$

$$Q_{ee,ii}(f_{e,i})(v) := \nu_{ee,ii} \left(\mathcal{M}_{n_{e,i}, \mathbf{u}_{e,i}, T_{e,i}} - f_{e,i} \right) ,$$

$$Q_{ei}(f_e, f_i)(v) := \nu_{ei} \left(\mathcal{M}^e_{n_e, \mathbf{u}_i, T_i} - f_e \right) , \quad Q_{ie}(f_i, f_e)(v) := \nu_{ie} \left(\mathcal{M}^i_{n_i, \mathbf{u}_e, T_e} - f_i \right) ,$$

Aim:

- find the right scaling, leading to the electron Boltzmann relation
- design an AP-scheme, allowing to follow this kinetic-adiabatic transition
- derive closure rel. for a fluid model describing the electr. dyn. close to adiabatic regime

Different asymptotics

- Asymptotics : Diff. scalings of the electr. Boltzmann eq.

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon^\alpha}\mathbf{v}\cdot\nabla_x f^\varepsilon - \frac{1}{\varepsilon^\beta}\left(\mathbf{E} + \frac{1}{\varepsilon^\gamma}\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v f^\varepsilon = \frac{1}{\varepsilon^\delta}Q(f^\varepsilon)\,,$$

lead to diff. macroscopic regimes in the limit $\varepsilon \to 0$:

➠ Hydrodynamic limit: $\alpha = \beta = \gamma = 0, \quad \delta = 1$

➠ Drift-Diffusion limit: $\alpha = \beta = 1, \quad \gamma = 0, \quad \delta = 2$

- Question : Which scaling leads to the Boltzmann relation:

$$n(t,\mathbf{x}) = c(t,\mathbf{x}_\perp)\,\exp\left(\frac{e\phi(t,\mathbf{x})}{k_B T(t,\mathbf{x}_\perp)}\right)\,, \quad \mathbf{x} = (\mathbf{x}_\perp, x_\parallel) \in \mathbb{R}^3$$

- Signification : quasi-instantaneous adjustment of the electron density n to any kind of potential perturbation
- Use : Approx. for electron dyn. in ion turbulence simulations; Permits to avoid time step restriction due to fast electron dynamics.

AP-scheme for the 1D electron kinetic equation in the adiabatic regime

Two ingredients:

- Micro-Macro decomposition (kinetic-fluid separation)
- Projection technique to enforce the adiabatic constraint

Work based on:

[2] A. De Cecco, C. Negulescu, S. Possanner "Asymptotic transition from kinetic to adiabatic electrons along magnetic field lines", to appear in SIAM MMS.

Electron dynamics

- Starting point: rescaled electron kinetic equation:

$$\partial_t f_e^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\, v\, \partial_x f_e^\varepsilon - \frac{1}{\sqrt{\varepsilon}}\, E\, \partial_v f_e^\varepsilon = \frac{1}{\varepsilon}\, [\mathcal{M}_{n_e^\varepsilon, \sqrt{\varepsilon} u_e^\varepsilon} - f_e^\varepsilon]\,,$$

$$n^\varepsilon(t,x) = \int_{\mathbb{R}} f^\varepsilon(t,x,v)\, dv\,, \quad \sqrt{\varepsilon}\, n^\varepsilon\, u^\varepsilon(t,x) = \int_{\mathbb{R}} v\, f^\varepsilon(t,x,v)\, dv\,.$$

- Adiabatic limit-regime for $\varepsilon \to 0$:

$$(L) \quad \begin{cases} \partial_t\, n_e^0 + \partial_x (n_e^0\, u_e^0) = 0 \\ \partial_x n_e^0 + E^0\, n_e^0 = 0\,. \end{cases}$$

➠ Electron Boltzmann relation:

$$n^0(t,x) = c(t)\, e^{\phi^0(t,x)}\,, \quad \forall (t,x) \in \mathbb{R}^+ \times [0,L]$$

➠ Determination of the constant $c(t)$ (periodic BC):

$$\partial_t\, \overline{n^0} = 0\,, \qquad \overline{n^0}(t) := \frac{1}{L}\int_0^L n^0(t,x)\, dx \;\Rightarrow\; \overline{n^0}(t) = \overline{n^0(0,\cdot)} = \overline{n_0^0}\,.$$

Simulations

- Boundary conditions: periodic in space on $I := (0, L)$ and

$$\lim_{v \to \pm\infty} f(t,x,v) = 0\,, \quad \forall t \in \mathbb{R}^+\,, \ \forall x \in [0,L]$$

- Electrostatic potential

$$\phi(x) = \cos(2\pi x) \qquad \Longrightarrow \qquad E(x) = -2\pi \sin(2\pi x) \quad \forall k \in \mathbb{N}$$

- Initial condition

$$f_0(x,v) = \frac{n_0(x)}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right)\,,$$

with

$$n_0(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-0.5)^2}{2\sigma^2}\right)\,,$$

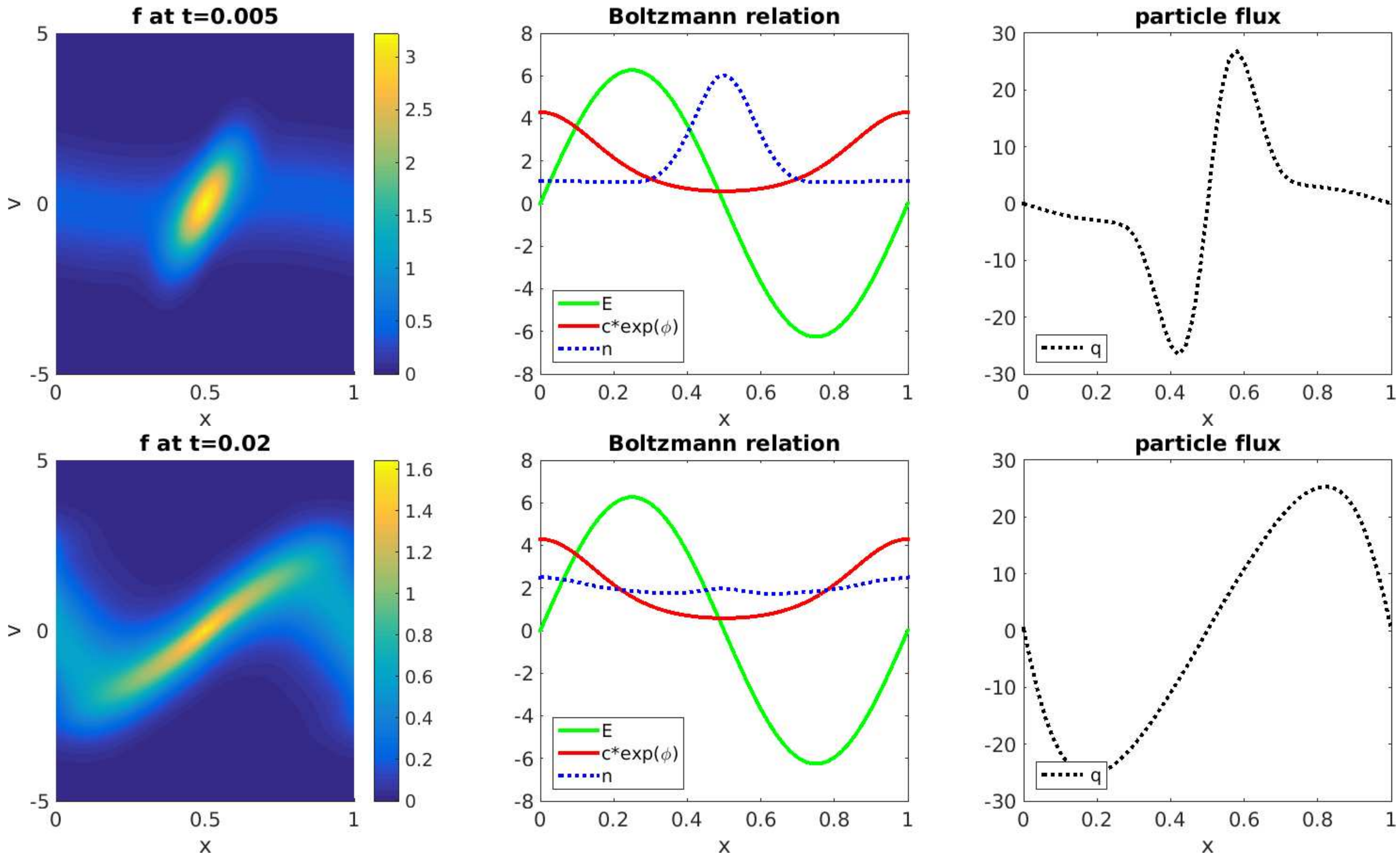
- Parameters

$$L = 1\,, \quad v_{min/max} = \pm 5\,, \quad T = 0.1\,, \quad \sigma = 0.5$$

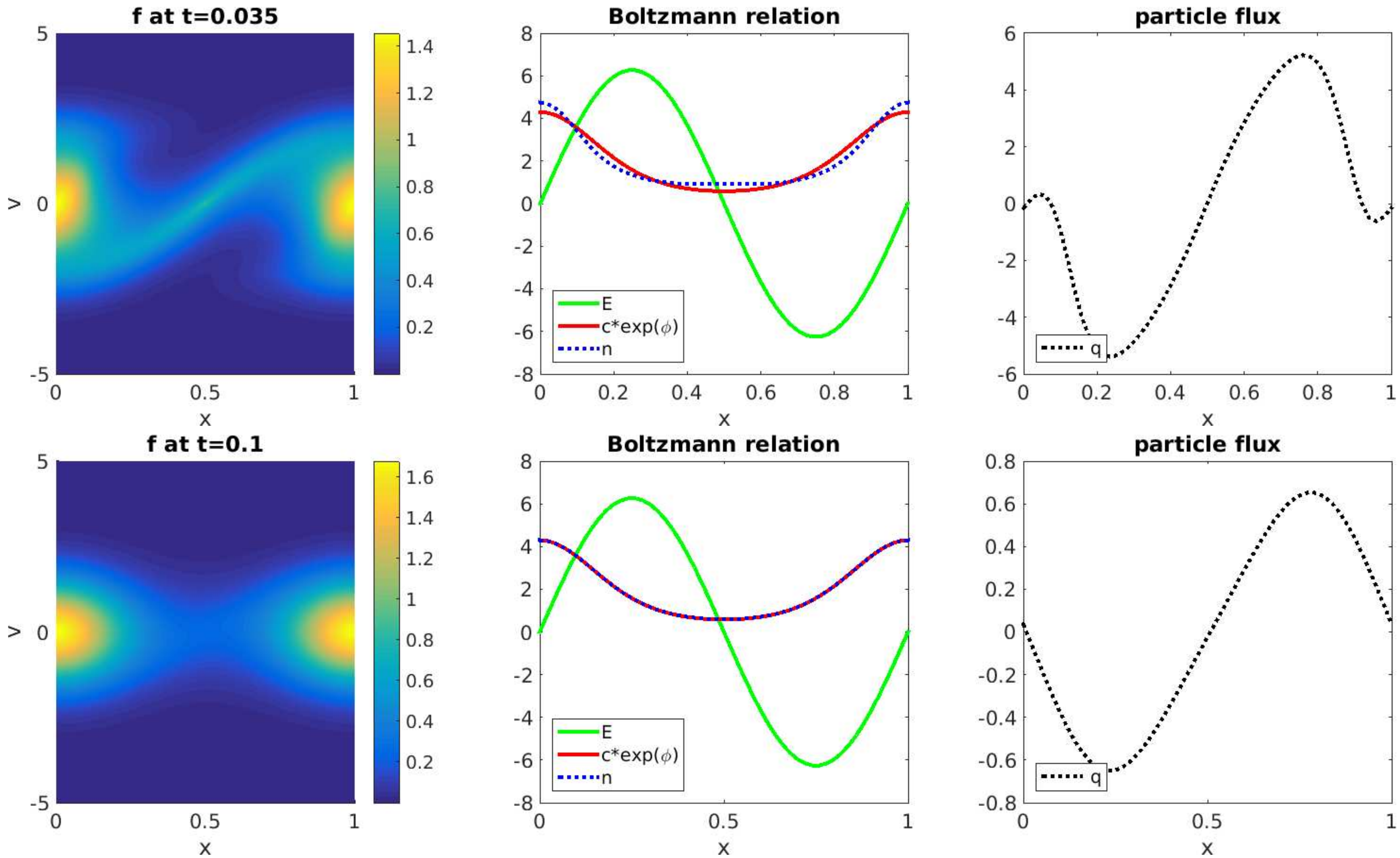
- Boltzmann relation

$$n(t/\varepsilon \to \infty, x) = c\, e^{\phi(x)}\,, \qquad c = \frac{\int_0^1 n_0\,dx}{\int_0^1 e^{\phi}\,dx}$$

Numerical results



Numerical results



Ion kinetic equation in the gyrokinetic regime

Work based on:

[3] B. Fedele, C. Negulescu, "Numerical study of an anisotropic Vlasov equation arising in plasma physics", submitted.

Ion dynamics

- Starting point: rescaled electron kinetic equation:

$$\partial_t f + \mathbf{v} \cdot \nabla_x f + \left[\mathbf{E} + \frac{1}{\epsilon} (\mathbf{v} \times \mathbf{B}) \right] \cdot \nabla_v f = 0$$

- Simplified toy model:

$$\partial_t f + \frac{v_y}{\epsilon} \partial_{v_x} f - \frac{v_x}{\epsilon} \partial_{v_y} f = 0$$

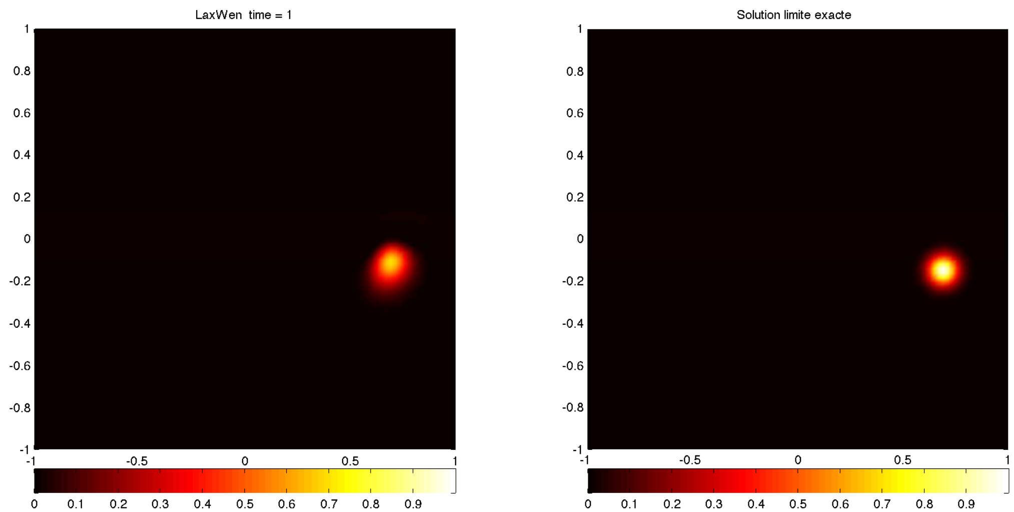
- Micro-Macro-Lagrangian via standrad method:

$$La)_\epsilon \quad \begin{cases} \dfrac{f^{\epsilon,n+1} - f^{\epsilon,n}}{\Delta t} + y\, \partial_x q^{\epsilon,n+1} - x\, \partial_y q^{\epsilon,n+1} = 0, \\ y\, \partial_x f^{\epsilon,n+1} - x\, \partial_y f^{\epsilon,n+1} = \epsilon \Big(y\, \partial_x q^{\epsilon,n+1} - x\, \partial_y q^{\epsilon,n+1} \Big) + (\Delta x \Delta y)^\gamma\, q^{\epsilon,n+1} \end{cases}$$

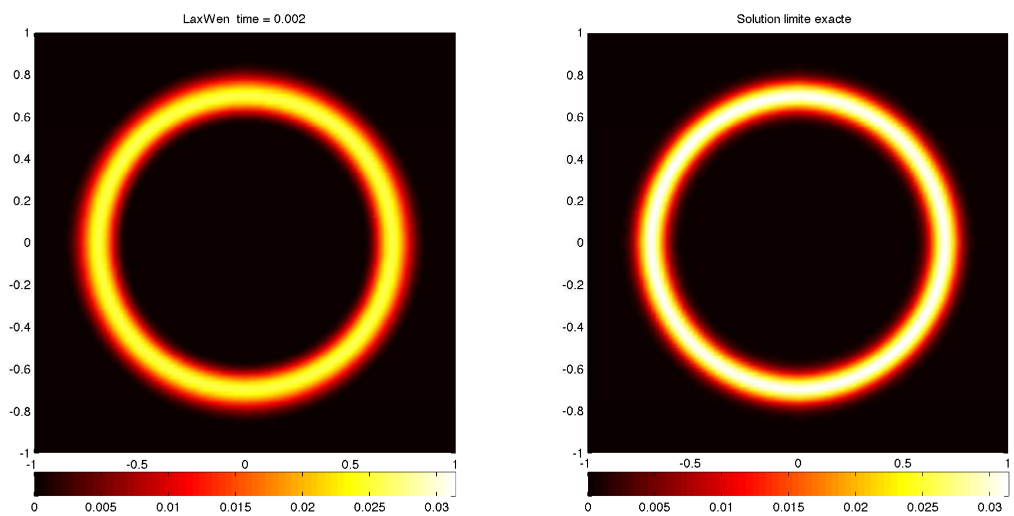
$$(IMEX)_\epsilon \quad \frac{f^{\epsilon,n+1} - f^{\epsilon,n}}{\Delta t} + \frac{y}{\epsilon}\, \partial_x f^{\epsilon,n+1} - \frac{x}{\epsilon}\, \partial_y f^{\epsilon,n+1} = 0.$$

Simulations, Micro-Macro scheme

Left: Num. sol.; Right: Exact sol.



$\varepsilon = 1, \; T = 1$



$\varepsilon = 10^{-10}, \; T = 2\,\Delta t$

Conclusions

- Singularly perturbed problems:
  - ➠ contain small parameters, that lead to various asymptotic regimes
  - ➠ classical schemes become too expensive, and even "unusable" in the limit regime
- Asymptotic-Preserving methodology:
  - ➠ offers simple, robust and efficient num. meth. for large class of singularly perturbed pb.
  - ➠ preserves at discrete level the limit asymptotics
  - ➠ solves the microscale, and automatically switches to a macroscopic solver for the limit pb.

Thank's